NOTE ON A CLASS OF TRANSFORMATIONS WHICH SURFACES MAY UNDERGO IN SPACE OF MORE THAN THREE DIMENSIONS.

By Simon Newcomb.

If the material bodies which surround us were placed in a space of more than three dimensions, their kinematic susceptibilities would be increased in a manner which, at first sight, would seem very extraordinary. Each body would, in fact, be susceptible of n independent forward motions, and $\frac{n(n-1)}{2}$ separate rotations, n being the number of dimensions of the space. My present purpose is not, however, to discuss the general theory of the subject, but to point out a special case of it as seen in a remarkable transformation to which closed surfaces may be subjected in space of four dimensions. The proposition in question may be expressed as follows:

If a fourth dimension were added to space, a closed material surface (or shell) could be turned inside out by simple flexure; without either stretching or tearing.

For simplicity we may suppose the surface to be spherical. Let

$$x, \; y, \; z, \; u,$$

be the general rectangular coördinates in the supposed space of four dimensions. An infinite plane space of three dimensions may then be represented by the equation

$$ax + by + cz + du = A,$$

a, b, etc., being any constants whatever. For simplicity we may suppose a, b and c all equal to zero, and the axes of x, y and z therefore to lie in the space of three dimensions under consideration. An Euclidian or natural space may then be represented by the single equation $u = A$, A being an arbitrary constant. The four-dimensional space may be divided into an infinity of Euclidian spaces by giving all possible values to A.

To define a surface in four-dimensional space by rectangular coördinates, two equations are necessary. If the surface is one which can exist in three-dimensional space, one of these equations must be of the first degree in x, y, etc. For example, the most general equations of the sphere in four-dimensional space are

$$(x-a)^2 + (y-b)^2 + (z-c)^2 + (u-d)^2 = r^2;$$

$$\alpha x + \beta y + \gamma z + \delta u = k, \tag{1}$$

α, β, γ, δ and k being constants, and a, b, c, d, the coördinates of the centre. This centre is not necessarily situated in the same Euclidian space with the surface; in fact there is a series of points, each of which is equidistant from every point of the surface, but only one of them lies in the same Euclidian space with the surface. This point is one whose coördinates fulfil the condition

$$\alpha a + \beta b + \gamma c + \delta d = k,$$

or,

$$\alpha(x-a) + \beta(y-b) + \gamma(z-c) + \delta(u-d) = 0.$$

If we choose our axes of coördinates so that the equations shall have the simplest forms, putting

$$\alpha = \beta = \gamma = 0, \quad \delta = 1,$$

the general equations (1) will become

$$(x-a)^2 + (y-b)^2 + (z-c)^2 + (u-d)^2 = r^2;$$

$$u = k.$$

Now, to consider the transformation of a material spherical surface, we must consider this surface as an indefinitely thin shell, situated between two surfaces. We shall suppose the natural space in which the sphere is situated in the beginning to be conditioned by the equation

$$u = k = 0,$$

and we shall take the centre of the sphere as the origin of coördinates. The equations of the inner surface of the spherical shell, which we may call A, will then be of the form

$$x^2 + y^2 + z^2 = r^2; \quad u = 0,$$

and of the outer one, B

$$x_1^2 + y_1^2 + z_1^2 = r_1^2; \quad u = 0.$$

Now suppose that, the outer surface remaining fixed, we move the inner one in the direction of the axis of u by a small quantity k, allowing its radius

at the same time to vary in such a way that the thickness of the shell shall remain unchanged. Its equations may then be expressed in the form

$$x^2 + y^2 + z^2 + u^2 = (r + \delta r)^2 ; \; u = k,$$

and we must, if possible, determine δr so that the thickness of the shell shall remain unaltered. To find the new thickness, we remark that the square of the distance of any point of the outer from any point of the inner surface is

$$(2) \qquad (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + u^2 .$$

The condition that this shall be a minimum for a given point of the outer surface is

$$(x - x_1)\, dx + (y - y_1)\, dy + (z - z_1)\, dz = 0,$$

dx, dy and dz being subject to the condition

$$xdx + ydy + zdz = 0.$$

The simultaneous existence of these equations depends upon our having

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} ,$$

or,

$$(3) \qquad x = \lambda x_1 , \quad y = \lambda y_1 , \quad z = \lambda z_1 ,$$

λ being a quantity which differs very little from unity. These values of x, y and z, being substituted in (2), give for the square of the thickness of the shell after the motion of the inner surface

$$(\lambda - 1)^2 (x_1^2 + y_1^2 + z_1^2) + k^2 = (\lambda - 1)^2 r_1^2 + k^2 .$$

Let us put $h = r_1 - r$, for the original thickness of the shell. In order that the thickness may remain unaltered, it is necessary and sufficient that we determine k and λ simultaneously in terms of an arbitrary angle θ by the conditions

$$(4) \qquad k = h \sin \theta ; \; 1 - \lambda = \frac{h}{r_1} \cos \theta .$$

The original position of the inner surface will be that corresponding to $\theta = 0$. Suppose, now, that θ increases from 0° to 180°, k and λ being constantly determined by the condition (4). It is then evident that the shell will experience no other deformation than that arising from flexure, the flexure involving a stretching of the outer surface which may be made indefinitely small by diminishing the thickness of the shell. When θ reaches 180° we shall once more have $k = 0$, so that the surface A will be brought back into

its original natural or Euclidian space. Moreover, λ being then greater than unity, the equations (3) show that the radius of this surface A, or $x^2 + y^2 + z^2$, will be greater than that of B.

The outer surface $x_1^2 + y_1^2 + z_1^2 = r_1^2$ will therefore be the inner one, and the other the outer one, the change being brought about by flexure alone. By the motion we have supposed, every point of the surface A has described a semicircle round the corresponding point of B, θ representing the angle of position of the line joining the two points during the motion. For simplicity we have supposed the surface A only to vary, the flexure taking place round B, but we might equally have supposed a mean surface to remain constant.